The six exponentials theorem in characteristic p

Dedicated to Professor Tosiro Tsuzuku on his sixtieth birthday

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1. Introduction

Using Schneider's method S. Lang [5] proved the following six exponentials theorem. Let α_1, α_2 be complex numbers, linearly independent over the rationals, and let β_1, β_2, β_3 be complex numbers, linearly independent over the rationals. Then at least one of the six numbers

$$e^{\alpha_\lambda \beta_\mu}(\lambda=1, 2, \mu=1, 2, 3)$$

is transcendental over the rationals.

In this note we consider the analogous problem for the Carlitz-ψ-function which can be regarded as the characteristic-p exponential function. For Carlitz-ψ-function we refer to L. Carlitz [1], L.I. Wade [8] and J. M. Geijsel [4]

Let $\mathbf{F}_p$ be the finite field of $q=p^f$ elements, p a prime and f a positive integer. Let x be an indeterminate and let $\mathbf{F}_q[x]$ denote the ring of polinomials in x with coefficients in $\mathbf{F}_q$. Let $\mathbf{F}_q(x)$ denote the field of rational functions in x with coefficients in $\mathbf{F}_q$. For $\alpha \in \mathbf{F}_q(x)$, there exist E, $G \in F_q[x]$ with $G \neq 0$ such that $\alpha = E/G$. We define dg α by

$$\begin{aligned} \mathrm{dg}\,\alpha &= -\infty && \text{if } \alpha=0, \\ &= \deg E - \deg \mathrm{G} && \text{if } \alpha \neq 0. \end{aligned}$$

With this definition, it is clear that $\alpha \longmapsto \mathrm{dg}\,\alpha$ is a (logarithmic) non-archimedean valuation on $\mathbf{F}_q(x)$. The completion of $\mathbf{F}_q(x)$ under this valuation is the formal Laurent-series field $\mathbf{F}_q((x^{-1}))$. Let Φ be the completion of the algebraic closure of $\mathbf{F}_q((x^{-1}))$. Then the field Φ is algebraically closed, and the valuation dg can be extended to Φ in a unique way (see [4], §1).

The Carlitz-ψ-function is defined by

$$\psi(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{F_k} \mathrm{t}^{q^k},$$

where

$$F_k = \prod_{j=0}^{k-1} (x^{q^k} - x^{q^j}).$$

The fuction $\psi(t)$ is an entire function, i.e. $\psi(t)$ cnverges for all $\mathrm{t} \in \Phi$, and can be written as an infinite product

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