

On Ultraproducts of Finite Fields

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Abstract

We treat an application of nonstandard models of finite fields to field theory and diophantine equations. We can investigate a sequence of diophantine equations as one diophantine equation in nonstandard model. By this principle, we can prove the existence of interesting family of diophantine equations having no integer solutions.

We believe that mathematical logic must be an essential tool to investigate mathematics. It is well known that ultraproducts of finite fields are of characteristic zero. But this is very interesting fact, for it seems to mean that finite fields contain infinite fields as subfield in some sense. This is paradoxical, therefore active to deduce many applications. In this paper, we write down a few of these.

1. Basic facts

The existence of nonprincipal ultrafilters of N is proved by axiom of choice. Throughout this paper we fix a nonprincipal ultrafilter \mathcal{U} of N . We let denote *N , *Z as ultrapowers of N , Z with respect to \mathcal{U} respectively. Let ${}^*n \in {}^*N$, then there is a sequence $(n_i : i \in N)$ such that

$${}^*n = {}^*(n_i : i \in N) : \text{equivalence class of } (n_i : i \in N) \text{ modulo } \mathcal{U}.$$

We let denote ${}^*Z/{}^*n{}^*Z$ as the ultraproduct of rings $(Z/n_iZ : i \in N)$ with respect to \mathcal{U} . If ${}^*p \in {}^*N$ is prime, then ${}^*Z/{}^*p{}^*Z$ is a field, so we denote ${}^*F_{*p}$ as ${}^*Z/{}^*p{}^*Z$. Let ${}^*p \in {}^*N$ be prime. For ${}^*n \in {}^*N$ and ${}^*f \in {}^*F_{*p}$, we can define ${}^*n{}^*f$ and ${}^*f{}^*n$. Suppose ${}^*n = {}^*(n_i : i \in N)$ and ${}^*f = {}^*(f_i : i \in N)$, then we define

$${}^*n{}^*f = {}^*(n_i f_i : i \in N), \quad {}^*f{}^*n = {}^*(f_i n_i : i \in N).$$

Moreover we can define $\sum_{j=1}^{*n} {}^*f_j$ and $\prod_{j=1}^{*n} {}^*f_j$ for ${}^*f_j \in {}^*F_{*p}$.

Let ${}^*f_j = {}^*(f_j^{(i)} : i \in N)$, then we define

$$\sum_{j=1}^{*n} {}^*f_j = {}^*(\sum_{j=1}^{n_i} f_j^{(i)} : i \in N)$$

and

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