A Note on a Fibre Metric in a Tangent Bundle: On a generalization of the Riemannian space

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Abstract

The similarity and the difference between the usual Riemnnian metric and a general fibre-metric in a tangent bundle are pointed out. The position-dependent inner product is introduced. The theory of the L-Riemannian space is given. The notion of the space of constant curvature in the Riemannian geometry is generalized and the properties of the space of constant s-L-curvature are given.

§1. Introduction

In this note we point out the similarity and the difference between the usual Riemannian metric and a general fibre-metric in a tangent bundle. (See Kobayahi-Nomizu[2], p. 116. Hereafter to be referred to as K-N[2] unless we specially note.)

Let M denote a paracompact C^∞-manifold with dim $M=n$. A fibre metric g in a tangent buddle $E=T(M)$ (to be referred simply as the fibre metric g) is an assignment, to each $x\in M$, of an inner product g_x in the fibre $\pi_E^{-1}(x)=T_x(M)$, which is differentiable in x in the sence that, if ϕ and ψ are differentiable cross sections of $T(M)$, then $g_x(\phi(x), \psi(x))$ depends differentiably on x. Here it shoud be noted that the inner product g_x depends on th fibre $T_x(M)$ at each $x\in M$. In this sense we might call g_x as the "position-dependent inner product." On the other hand the components of the usual Riemannian metric G is defined by an inner product in a single fibre $T_x(M)$ at a fixed point $x\in M$:

$$G_{ij}=\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$$

in terms of a natural frame $((X_1)_x, \ldots, (X_n)_x)$, $X_i=\partial/\partial x^i$ at $x\in M$ where the inner product $(\ ,\ )$ in $T_x(M)$ is regarded as an inner product in a single vector space even if the point $x\in M$ changes. (cf. Akizuki[1]). In this connection we express the components of the fibre metric g with respect to the local coordinates as follows:

$$g_{ij}(x)=\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)_x,$$

where the inner product $(\ ,\ )_x$ is taken in each fibre $T_x(M)$ depending on the point $x\in M$. The fibre metric is adapted to the full bundle $L(M)$ of linear frames over M just as the usual Riemannian metric is adapted to the reduced bundle $O(M)$ of orthonormal frames over M.

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