On a Generalization of Hilbert Transform

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Abstract

Hilbert Transform is generalized based on the assumption of the complex analytic functions only in the strict upper-half plane $Imz>0$, insted of the usual assumption in $Imz \geqq 0$. Our method gives the mathematical basis of the regularization of some singular integrals and also gives a new viewpoint of distribution theory and hyperfunction theory. Our viewpoint is excelent because it is easy to understand and is explicit in presentation: we do not use the concept of "the test function", "the inductive limit" and "the defining function". The mathematical connection with two dimensional potential-theory is also given. It is interesting to apply our result to the global theory of the electric heart pulse of biologycal systems.

1. INTRODUCTION

Hibert Transform (to be called simply HT hereafter) [1] is one of integral transforms. It is used in Kramers-Kronig relations in solid state optics and in dispersion relations in high energy physics. Its derivation is based on the complex analytic functions in upper half plane ($Imz \geqq 0$) and on Cauchy's Integral formula. Our present theory uses the complex analytic function $f(z)$ in the strict upper half plane ($Imz>0$) and the complex analytic function $g(z)$ in the strict lower half plane ($Imz<0$). Considering both $f(z)$ and $g(z)$, we obtain a regularization of singular integrals and the connection with generalized functions [2](and hyperfunctions [3]). We can obtain explicit presentations and a new standpoint.

2. THE BOUNDEDNESS CONDITION IN THE STRICT UPPER HALF PLANE

Let $f(z)$ be analytic in the strict upper half plane ($Imz>0$). For ζ inside the region of the closed integral-path C_ε, we have the equation

$$f(\zeta)=\frac{1}{2\pi i}\oint_{C_\varepsilon} f(z)\left[\frac{1}{z-\zeta}-\frac{1}{z-\bar{\zeta}}\right]dz$$

Next we use the boundedness condition: For any $\delta>0$, there exists a constant $M>0$, such that $|f(z)|<M$ in $Imz>\delta$. We devide the integral and take the limit $R\to\infty$ to give

$$f(\zeta)=\frac{1}{2\pi i}\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} f(z)\left[\frac{1}{z-\zeta}-\frac{1}{z-\bar{\zeta}}\right]dz$$

Thus we have

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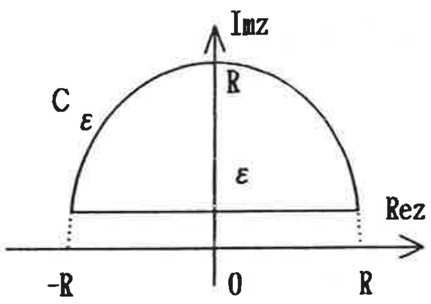

$$f(\xi+i\eta)=\frac{\eta}{\pi}\int_{-\infty}^{\infty}\frac{f(x+i\varepsilon)}{(x-\xi+i\varepsilon)^2+\eta^2}dx,\ (0<\varepsilon<\eta)$$

Here the right hand side does not depend on ε. Let $g(z)$ be analytic in the strict lower half plane ($Imz<0$). For any δ, let there exist a connstant M' such that $|g(z)|<M'$ in $Imz<\delta$. Then we have

$$g(\xi-i\eta)=\frac{\eta}{\pi}\int_{-\infty}^{\infty}\frac{g(x-i\varepsilon)}{(x-\xi-i\varepsilon)^2+\eta^2}dx,\ (0<\varepsilon<\eta)$$

Here again the right hand side does not depend on ε. For $f(z)$ and $g(z)$ above, we have the equation

$$f(\xi+i\eta)-g(\xi-i\eta)=\frac{\eta}{\pi}\int_{-\infty}^{\infty}dx\left[\frac{f(x+i\varepsilon)}{(x-\xi+i\varepsilon)^2+\eta^2}-\frac{g(x-i\varepsilon)}{(x-\xi-i\varepsilon)^2+\eta^2}\right]\quad (\varepsilon\text{-indep., } 0<\varepsilon<\eta)$$

Specially, if $f(z)$ is bounded in both $Imz>\delta$ and $Imz<\delta$, then we have the equation

$$f(\xi+i\eta)-f(\xi-i\eta)=\frac{\eta}{\pi}\iint_{-\infty}^{\infty}\frac{f(x)}{(x-\xi)^2+\eta^2}dx$$
$$\equiv\frac{\eta}{\pi}\left(\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon}+\int_{\infty-i\varepsilon}^{-\infty-i\varepsilon}\right)\frac{f(z)}{(z-\xi)^2+\eta^2}dz,$$
$$(\varepsilon\text{-indep., } 0<\varepsilon<\eta)$$

In the usual HT we know the equation,

$$f(\xi+i\eta)=\frac{\eta}{\pi}\int_{-\infty}^{\infty}\frac{f(x)}{(x-\xi)^2+\eta^2}dx$$

If we achieve the formal replacement,

$$\int_{-\infty}^{\infty}\cdots dx \quad\rightarrow\quad \iint_{-\infty}^{\infty}\cdots dx$$

then the above usual equation in HT simply gives us our new equation for $f(z)$. Our new integral is to be called "the double trip integral" or simply DTI hereafter. Thus we have a generalization (to be called simply GHT hereafter) of the equation in the usual HT. The above replacement is important for the reguralization of singular real integrals in our theory. On the other hand we have the definition of "the vertical differentiation VD" or "the imaginary differentiation ID" of the real-variable function $f(\xi)$.

$$\frac{vd}{vd\xi}f(\xi)\equiv\lim_{\eta\to+0}\frac{1}{2i\eta}\left\{f(\xi+i\eta)-f(\xi-i\eta)\right\}$$
$$=\frac{1}{2\pi i}\lim_{\eta\to+0}\iint_{-\infty}^{\infty}\frac{f(x)}{(x-\xi)^2+\eta^2}dx$$

Here it is to be noted that we do not assume both

1) the standpoint of hyperfunction,

and

2) Lippmann-Schwinger formula [4], i.e.,

$$\frac{1}{x \pm i0} = P\frac{1}{x} \mp i\pi\delta(x)$$

3. THE BOUNDEDNESS CONDITION OF $zf(z)$ IN $Imz>0$

Let $f(z)$ be analytic in $Imz>0$. And let $|zf(z)|<K$ in $Imz>\delta$ for any constant $\delta>0$. Similarly as in the section 2, we have the equation,

$$f(\zeta) = \frac{1}{2\pi i}\oint_{C_\varepsilon} f(z)\left[\frac{1}{z-\zeta} + \frac{1}{z-\bar{\zeta}}\right]dz$$

Our boundedness condition enables us to obtain the integral representation of $f(\xi+i\eta)$,

$$f(\xi+i\eta) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{x-\xi+i\varepsilon}{(x-\xi+i\varepsilon)^2+\eta^2} f(x+i\varepsilon)dx,$$

(ε-indep., $0<\varepsilon<\eta$)

We devide the complex valued function $f(x+i\varepsilon)$ into the real part $u(x, \varepsilon)$ and the imaginary part $v(x, \varepsilon)$. We have the representation for u,

$$u(\xi, \eta) = \frac{1}{\pi}\int_{-\infty}^{\infty} Im\left[\frac{x-\xi+i\varepsilon}{(x-\xi+i\varepsilon)^2+\eta^2} f(x+i\varepsilon)\right]dx$$

($\varepsilon-$indep., $0<\varepsilon<\eta$)

If we take the limit $\varepsilon \to +0$ in the integrand, then we have the relation between $u(\xi, \eta)$ and $v(x, 0)$,

$$u(\xi, \eta) \sim \frac{1}{\pi}\int_{-\infty}^{\infty} dx \frac{(x-\xi)v(x, 0)}{(x-\xi)^2+\eta^2}, \qquad (\varepsilon \to +0)$$

Taking the limit $\eta \to +0$ in the integrand, we have the relation between $u(\xi, 0)$ and $v(x, 0)$, which is usually interpreted as Cauchy's principal value integral in HT :

$$u(\xi, 0) \sim \frac{1}{\pi}\int_{-\infty}^{\infty} dx \frac{v(x, 0)}{x-\xi}, \qquad (\eta \to +0)$$

$$\sim \frac{1}{\pi} P\int_{-\infty}^{\infty} dx \frac{v(x, 0)}{x-\xi}, \qquad (\mathrm{HT})$$

For Imaginary part v we have the representation,

$$v(\xi, \eta) = -\frac{1}{\pi}\int_{-\infty}^{\infty} Re\left[\frac{x-\xi+i\varepsilon}{(x-\xi+i\varepsilon)^2+\eta^2} f(x+i\varepsilon)\right]dx,$$

(ε-indep., $0<\varepsilon<\eta$)

Taking the limit $\varepsilon \to +0$ in the integrand, we have the relation between $v(\xi, \eta)$ and $u(x, 0)$,

$$v(\xi, \eta) \sim -\frac{1}{\pi}\int_{-\infty}^{\infty} dx \frac{(x-\xi)u(x, 0)}{(x-\xi)^2+\eta^2}, \qquad (\varepsilon \to +0)$$

Taking the limit $\eta \to +0$ in the integrand, we have the relation between $u(\xi, 0)$ and $v(x, 0)$, which is usually interpreted as Cauchy's principal value integral in HT :

$$v(\xi,0)\sim-\frac{1}{\pi}\int_{-\infty}^{\infty}dx\,\frac{u(x,0)}{x-\xi},\qquad(\eta\to+0)$$

$$\sim-\frac{1}{\pi}P\int_{-\infty}^{\infty}dx\,\frac{u(x,0)}{x-\xi},\qquad\text{(HT)}$$

From our standpoint the usual HT gives the relations between the real part $u(x,0)$ and the imaginary part $v(x,0)$ of the analytic function $f(z)$ in $Imz\geqq 0$. If $f(z)$ is a real function, we have the identity

$$v(x,0)=0,\qquad\text{(HT, the real function)}$$

Let $g(z)$ be analytic in Imz<0. And let $|zg(z)|<K'$ in Imz$<-\delta$ for any constant $\delta>0$. Then we have the equation,

$$g(\xi-i\eta)=-\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{x-\xi-i\varepsilon}{(x-\xi-i\varepsilon)^2+\eta^2}g(x-i\varepsilon)dx$$

$$(\varepsilon\text{-indep., }0<\varepsilon<\eta)$$

Therefore we have

$$f(\xi+i\eta)+g(\xi-i\eta)$$
$$=\frac{1}{\pi i}\int_{-\infty}^{\infty}dx\left[\frac{(x-\xi+i\varepsilon)f(x+i\varepsilon)}{(x-\xi+i\varepsilon)^2+\eta^2}-\frac{(x-\xi-i\varepsilon)g(x-i\varepsilon)}{(x-\xi-i\varepsilon)^2+\eta^2}\right]$$

$$(\varepsilon\text{-indep., }0<\varepsilon<\eta)$$

Specially if $f(z)$ satisfies the same conditions in both Imz$>\delta$ and Imz$<-\delta'$, we have

$$f(\xi+i\eta)+f(\xi-i\eta)=\frac{1}{\pi i}\iint_{-\infty}^{\infty}\frac{x-\xi}{(x-\xi)^2+\eta^2}f(x)dx$$

or

$$\frac{f(\xi+i\eta)+f(\xi-i\eta)}{2}=\frac{1}{2\pi i}\iint_{-\infty}^{\infty}dx\,\frac{x-\xi}{(x-\xi)^2+\eta^2}f(x)$$

In HT this eq. corresponds to

$$f(\xi+i\eta)=\frac{1}{\pi i}\int_{-\infty}^{\infty}dx\,\frac{(x-\xi)f(x)}{(x-\xi)^2+\eta^2},\qquad\text{(HT)}$$

Here the integral is the usual one on the real axis.
For example we consider

$$f(z)=\frac{1}{z}$$

We have

$$f(\xi+i\eta)-f(\xi-i\eta)=\frac{-2i\eta}{\xi^2+\eta^2}=\frac{\eta}{\pi}\iint_{-\infty}^{\infty}dx\,\frac{f(x)}{(x-\xi)^2+\eta^2}$$

$$=\frac{-2i\eta\varepsilon}{\pi}\int_{-\infty}^{\infty}\frac{1}{x^2+\varepsilon^2}g(x,\xi;\eta,\varepsilon)dx$$

where

$$g(x,\xi;\eta,\varepsilon)=\frac{(x-\xi)(3x-\xi)+\eta^2-\varepsilon^2}{\{(x-\xi)^2+\eta^2-\varepsilon^2\}^2+4\varepsilon^2(x-\xi)^2}$$

$$\lim_{x\to 0}\lim_{\varepsilon\to 0}g(x,\xi;\eta,\varepsilon)=\lim_{x\to 0}\frac{(x-\xi)(3x-\xi)+\eta^2}{\{(x-\xi)^2+\eta^2\}^2}=\frac{1}{\xi^2+\eta^2}$$

Therefore we have the relation

$$\frac{\varepsilon}{\pi}\int_{-\infty}^{\infty}\frac{1}{x^2+\varepsilon^2}g(x,\xi;\eta,\varepsilon)dx=\lim_{y\to 0}\lim_{\delta\to 0}g(y,\xi;\eta,\delta)$$

$$=\frac{1}{\xi^2+\eta^2}\qquad(\varepsilon\text{-indep., }0<\varepsilon<\eta)$$

It is interesting to compare this results with the property of δ-function,

$$\int_{-\infty}^{\infty}\delta(x)h(x)dx=h(0),\quad \frac{1}{\pi}\cdot\frac{\varepsilon}{x^2+\varepsilon^2}\sim\delta(x)$$

Coulomb potential $1/x$ and its Fourier transform, $1/(\xi^2+\eta^2)$, of the screened Coulomb (Yukawa) potential are related as follows:

$$f(x)=\frac{1}{x}\quad\to\quad F(\xi,\eta)=(Sf)(\xi,\eta)$$

$$\equiv\frac{1}{2i\eta}\{f(\xi+i\eta)-f(\xi-i\eta)\}=-\frac{1}{\xi^2+\eta^2}$$

Here the "screening operation" or "Yukawa operation" S is related to VD as follows:

$$\frac{vd}{vd\xi}f(\xi)=\lim_{\eta\to+0}\Big[(Sf)(\xi,\eta)\Big]$$

for general functions $f(x)$ beyond the function $1/x$. In other words, S is the complex difference-scheme for VD, similar to the difference-scheme for the real differentiation.
For $f(z)=1/z$ we have the relation,

$$f(\xi+i\eta)+f(\xi-i\eta)=\frac{2\xi}{\xi^2+\eta^2}=\frac{1}{\pi i}\iint_{-\infty}^{\infty}\frac{x-\xi}{(x-\xi)^2+\eta^2}f(x)dx$$

$$=\frac{2}{\pi}\int_{-\infty}^{\infty}dx\,\frac{\varepsilon}{x^2+\varepsilon^2}h(x,\xi;\eta,\varepsilon)$$

where

$$h(x,\xi;\eta,\varepsilon)=\frac{(\xi-2x)\{(x-\xi)^2+\varepsilon^2\}+\xi\eta^2}{\{(x-\xi)^2+\eta^2-\varepsilon^2\}^2+4\varepsilon^2(x-\xi)^2}$$

$$\lim_{x\to 0}\lim_{\varepsilon\to 0}h(x,\xi;\eta,\varepsilon)=\lim_{x\to 0}\frac{(\xi-2x)(x-\xi)^2+\xi\eta^2}{\{(x-\xi)^2+\eta^2\}^2}=\frac{\xi}{\xi^2+\eta^2}$$

Therefore we have the relation

$$\frac{\varepsilon}{\pi}\int_{-\infty}^{\infty}\frac{1}{x^2+\varepsilon^2}h(x,\xi;\eta,\varepsilon)dx$$
$$=\lim_{y\to 0}\lim_{\delta\to 0}h(y,\xi;\eta,\delta)=\frac{\xi}{\xi^2+\eta^2}$$

(ε-indep., $0<\varepsilon<\eta$)

Following examples shows the regularization of singular integrals by our approach:

$$\fint_0^1\frac{1}{x}dx\equiv\frac{1}{2}\lim_{\varepsilon\to+0}\left(\int_{0+i\varepsilon}^{1+i\varepsilon}+\int_{1-i\varepsilon}^{0-i\varepsilon}\right)\frac{1}{z}dz$$
$$\frac{1}{2}\lim_{\varepsilon\to+0}\left[\log\frac{x^2-\varepsilon^2+2i\varepsilon x}{x^2+\varepsilon^2}\right]_0^{x=1}=\frac{1}{2}\{\log 1-\log(-1)\}$$
$$=\frac{1}{2}\left\{2n\pi i-(2n+1)\pi i\right\}=-\frac{\pi i}{2}$$
$$\fint_{-1}^1\frac{1}{x}dx\equiv\frac{1}{2}\lim_{\varepsilon\to+0}\left(\int_{-1+i\varepsilon}^{1+i\varepsilon}+\int_{1-i\varepsilon}^{-1-i\varepsilon}\right)\frac{1}{z}dz$$
$$=\frac{1}{2}\lim_{\varepsilon\to 0}\left[\log\frac{x^2-\varepsilon^2+2i\varepsilon x}{x^2+\varepsilon^2}\right]_{-1}^{x=1}=\frac{1}{2}\log 1=0$$

(taking the principal value of log function)

Here it is to be noted that the integrals concerning δ-function,

$$\int_0^1\delta(x)dx,\quad \int_a^\infty\delta(x-a)dx$$

are undefined in the distribution theory and in the hyperfunction theory, as is well known.

4. GENERALIZED HILBERT TRANSFORM AND TWO DIMENSIONAL POTENTIAL

From our integral representation we have the imaginary part and the real part of $f(\xi+i\eta)$,

$$Im\ f(\xi+i\eta)=\frac{\eta}{2\pi i}\iint_{-\infty}^{\infty}\frac{f(x)}{(x-\xi)^2+\eta^2}dx$$
$$Re\ f(\xi+i\eta)=\frac{1}{2\pi i}\iint_{-\infty}^{\infty}\frac{x-\xi}{(x-\xi)^2+\eta^2}f(x)dx$$

Here our assumptions are:

1) $f(z)$ is analytic in $Imz\neq 0$
2) $|f(z)|<M$ in $Imz>\delta$ for any $\delta>0$
3) $|f(z)|<M'$ in $Imz<-\delta'$ for any $\delta'>0$
4) $\overline{f(\xi+i\eta)}=f(\xi-i\eta)$, "$\overline{\ \ }$" means the complex conjugation

For the imaginary part of $f(\xi+i\eta)$ we can rewrite as

$$Im\ f(\xi+i\eta)=\frac{1}{\pi i}\iint_{-\infty}^{\infty}\left\{\frac{\partial}{\partial\eta}\log[(x-\xi)^2+\eta^2]\right\}f(x)dx$$
$$\simeq\frac{1}{\pi i}\frac{\partial}{\partial\eta}\iint_{-\infty}^{\infty}f(x)\log[(x-\xi)^2+\eta^2]dx$$

Here in the last equation we assumed the order of η-differentiation and DTI with respect to x, where the similarity between our result and the logarithmic potential in the well known two dimentional potential-theory [5] is obvious. For the real part of $f(\xi+i\eta)$ we can rewrite as

$$Re\ f(\xi+i\eta)=\frac{1}{\pi i}\iint_{-\infty}^{\infty}\left\{\frac{\partial}{\partial x}\log\left[(x-\xi)^2+\eta^2\right]\right\}f(x)dx$$

Here the partial derivative with x in the integrand has the same form as the well known double-layer potential in two dimension [5]. For the analytic function $f(z)$ in the strict upper half plane with bounded condition for $zf(z)$ for the upper part of z-plane we have the expression

$$\begin{aligned} f(\xi+i\eta)&=\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{x-\xi+i\varepsilon}{(x-\xi+i\varepsilon)^2+\eta^2}f(x+i\varepsilon)dx \\ &=\frac{1}{\pi i}\lim_{\varepsilon\to+0}\int_{-\infty}^{\infty}\frac{x-\xi+i\varepsilon}{(x-\xi+i\varepsilon)^2+\eta^2}f(x+i\varepsilon)dx \\ &\simeq\frac{1}{\pi i}\lim_{\varepsilon\to+0}\int_{-\infty}^{\infty}\frac{x-\xi}{(x-\xi)^2+\eta^2}f(x+i\varepsilon)dx \end{aligned}$$

Here in the last step a suitable assumption for ε-dependence in the integrand was made. From this we have the representation for the real part $u(\xi,\eta)$ and the imaginary part $v(\xi,\eta)$:

$$\begin{aligned} u(\xi,\eta)&\simeq\frac{1}{\pi}\lim_{\varepsilon\to+0}\int_{-\infty}^{\infty}\frac{x-\xi}{(x-\xi)^2+\eta^2}v(x,\varepsilon)dx\equiv(Tv)(\xi,\eta) \\ &=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x-\xi}{(x-\xi)^2+\eta^2}\mu_v(x)dx, \\ &\mu_v(x)\equiv\operatorname*{Lim}_{\varepsilon\to+0}v(x,\varepsilon) \end{aligned}$$

and

$$\begin{aligned} v(\xi,\eta)&\simeq-\frac{1}{\pi}\lim_{\varepsilon\to+0}\int_{-\infty}^{\infty}\frac{x-\xi}{(x-\xi)^2+\eta^2}u(x,\varepsilon)dx=-(Tu)(\xi,\eta) \\ &=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x-\xi}{(x-\xi)^2+\eta^2}\mu_u(x)dx, \\ &\mu_u(x)=\operatorname*{Lim}_{\varepsilon\to+0}u(x,\varepsilon) \end{aligned}$$

Here the symbol Lim in the definition of μ_v and μ_u means the generalized limit in the similar sense of generalized functions. We have the equations for the operator T,

$$(T^2v)(x,\varepsilon)=[T(Tv)](x,\varepsilon)=-v(x,\varepsilon)$$
$$(T^2u)(x,\varepsilon)=[T(Tu)](x,\varepsilon)=-u(x,\varepsilon)$$

for the imaginary and real parts of any analytic-type function. Thus we have the operator equation for T:

$$T^2=-1 \quad \text{or} \quad T^2+1=0$$

These equations have the same form as in the usual HT, though our present result is a wide generalization of the usual HT. We call T as "the imaginary-unit type operator" or simply IUTO.

5. CONCLUSION AND OBSERVATION

We genaralized HT based on the complex function analytic only in $Imz>0$ different from HT which is founded on the complex function analytic in $Imz \geqq 0$. Our GHT can be expressed in the analogous form as HT using DTI, with which the reguralization of singular real integrals can be achieved. We naturaly introduced VD(ID), which is related with the screening operation S, which is the complex finite difference-scheme for VD. The logarithmic and the double-layer potentals in two dimension are connected with our GHT. The generalized limit Lim was introduced in the similar sense of generalized functions. The IUTO, the Imaginary-Unit Type Operators, was introduced, which will have important role in our succeeding work. The application of GHT to the electric heart pulse in biological systems [6] is the interesting problem.

6. ACKNOWLEDGEMENT

The present author would like to thank Professor Saburo Saito of Gunma Univ. for his interest and the helpful discussion with him. He also thanks to Professor T. Matsuzaka for giving him the oppotunity to publish the present work.

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*The present paper is a part of the invited one-hour lecture given at the 4th International Colloqium on Differential Equations at Plovdiv Technical Univ., Bulgaria, August, 1993.